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New aspects of integrability of force-free Duffing–van der Pol oscillator and related nonlinear systems

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Abstract

In this paper, we show that the force-free Duffing–van der Pol oscillator is completely integrable for a specific parametric choice. We derive a general solution for this parametric choice. Further, we describe a procedure to construct the transformation which removes the time-dependent part from the first integral and provide the general solution by quadrature. The procedure is shown to have a wider applicability through additional examples. We also show that through our method one can deduce linearizing transformations in a simple and straightforward way and illustrate it with a specific example.

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1. Introduction

One of the well-studied but still challenging equations in nonlinear dynamics is the Duffing–van der Pol oscillator equation [1]. Its autonomous version (force-free) is

$$\ddot{x} + (\alpha + \beta x^2)\dot{x} - \gamma x + x^3 = 0 \tag{1}$$

where an overdot denotes differentiation with respect to time and α, β and γ are arbitrary parameters. Equation (1) arises in a model describing the propagation of voltage pulses along a neuronal axon and has received a lot of attention recently by many authors. A vast amount of literature exists on this equation, for details see, for example, [1, 2] and references therein. In most of the works non-integrability properties have been considered primarily since the forced version of equation (1) exhibits a rich variety of bifurcations and chaotic phenomena. When $\beta = 0$, equation (1) becomes the (force-free) Duffing oscillator whose integrability and non-integrability properties are well known [1], while when the cubic term is absent it is the standard van der Pol equation.

As far as the integrability properties of equation (1) are concerned not much progress has been made mainly due to the fact that it does not pass the Painlevé test as it admits a movable algebraic branch point and a local Laurent expansion in the form [3]

$$x(t) = \sqrt{\frac{3}{2\beta}}\tau^{-\frac{1}{2}} + \sqrt{\frac{3}{2\beta}}\left(\frac{3}{2\beta} - \frac{\alpha}{2}\right)\tau^{\frac{1}{2}} + a_3\tau + \cdots \tag{2}$$

where $\tau = (t - t_0)$ and t_0 and a_3 are arbitrary constants. However, it is known that the system (1) admits nontrivial symmetries and a first integral for a specific parametric choice [4]

$$\alpha = \frac{4}{\beta} \qquad \gamma = -\frac{3}{\beta^2}. \tag{3}$$

The associated first integral reads

$$\mathrm{e}^{\frac{3}{\beta}t}\left(\dot{x} + \frac{1}{\beta}x + \frac{\beta}{3}x^3\right) = I. \tag{4}$$

Rewriting (4) we get

$$\dot{x} + \frac{1}{\beta}x + \frac{\beta}{3}x^3 = I\,\mathrm{e}^{-\frac{3}{\beta}t} \tag{5}$$

which is nothing but a special case of the Abel equation of the first kind [5]. However, it has not been explicitly integrated directly due to the explicit time-dependent part in the first integral and so the complete integrability of (1) for the choice (3) has remained unclear upto now. In this work, we integrate equation (1) explicitly for the parametric choice (3) and provide a general solution for the above specific parametric choice of the Duffing–van der Pol nonlinear oscillator, thereby establishing its complete integrability. Further, we show that our procedure has a wider applicability, including linearization, through additional examples.

2. Integration of Duffing–van de Pol oscillator for the specific parametric choice (3)

Rewriting (1) with the specific parametric choice (3) we get

$$\ddot{x} + \left(\frac{4}{\beta} + \beta x^2\right)\dot{x} + \frac{3}{\beta^2}x + x^3 = 0. \tag{6}$$

Now introducing a transformation

$$w = -x\,\mathrm{e}^{\frac{1}{\beta}t} \qquad z = \mathrm{e}^{-\frac{2}{\beta}t} \tag{7}$$

where w and z are new dependent and independent variables respectively, equation (6) can be transformed to

$$w'' - \frac{\beta^2}{2}w^2w' = 0 \tag{8}$$

where a prime denotes differentiation with respect to z. Equation (8) can be integrated trivially to yield

$$w' - \frac{\beta^2}{6}w^3 = I \tag{9}$$

where I is the integration constant. Equivalently, the transformation (7) reduces (4) to this form. Solving (9), we obtain

$$z - z_0 = \frac{a}{3I}\left[\frac{1}{2}\log\left(\frac{(w+a)^2}{w^2 - aw + a^2}\right) + \sqrt{3}\arctan\left(\frac{w\sqrt{3}}{2a - w}\right)\right] \tag{10}$$

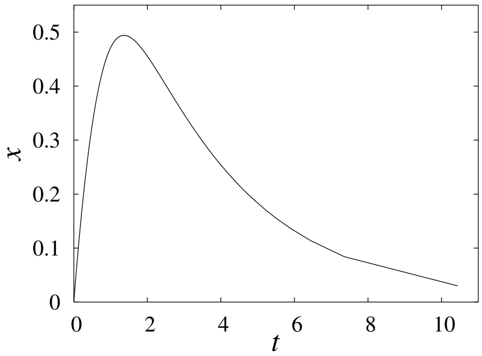

Figure 1. The solution plot of the Duffing–van der Pol oscillator equation (6) in the form (10) when $\beta = 3$.

where $a = \sqrt[3]{\frac{6I}{\beta^2}}$ and z_0 is the second integration constant (see figure 1 for a plot of the solution $x(t)$). Replacing the second term inside the square bracket in (10) in terms of a logarithmic function we get

$$\frac{1}{2}\left[\mathrm{i}\sqrt{3}\log\left(\frac{-2+\bar{w}+\mathrm{i}\sqrt{3}\bar{w}}{-2+\bar{w}-\mathrm{i}\sqrt{3}\bar{w}}\right)+\log\left(\frac{a(\bar{w}+1)^2}{(\bar{w}^2-\bar{w}+1)}\right)\right]=\frac{3I}{a}(z-z_0) \tag{11}$$

where $\bar{w} = \frac{w}{a}$. This can be further simplified to

$$\left[\frac{-2+\bar{w}(1+\mathrm{i}\sqrt{3})}{-2+\bar{w}(1-\mathrm{i}\sqrt{3})}\right]^{\mathrm{i}\sqrt{3}}\left[\frac{\bar{w}+1}{(\bar{w}^2-\bar{w}+1)}\right]=\frac{1}{a}\,\mathrm{e}^{\frac{6I}{a}(z-z_0)}. \tag{12}$$

Rewriting $\bar{w}$ and z in terms of old variables one can get the explicit solution for equation (6).

One may note that a simple solution for equation (6) can also be built from the first integral (9). Restricting $I = 0$ in equation (9) yields

$$w = \frac{\sqrt{3}}{\beta}(z_0 - z)^{-\frac{1}{2}}. \tag{13}$$

Rewriting (13) in terms of the old coordinates we get a particular solution for (6) in the form

$$x = -\frac{\sqrt{3}}{\beta\sqrt{\left(t_0\,\mathrm{e}^{\frac{2}{\beta}t}-1\right)}} \qquad t_0 > 1. \tag{14}$$

3. Method of constructing transformation

One can construct the transformation (7) systematically from the first integral (4). For example, let us split the functional form of the first integral I into two terms such that one involves all the variables $(t, x, \dot{x})$ while the other excludes $\dot{x}$, that is,

$$I = F_1(t, x, \dot{x}) + F_2(t, x). \tag{15}$$

Now let us split the function F_1 further into terms of two functions such that F_1 itself is a function of the product of the two functions, say, a perfect differentiable function $\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x)$ and another function $G_2(t, x, \dot{x})$, that is,

$$I = F_1\left(\frac{1}{G_2(t, x, \dot{x})}\frac{\mathrm{d}}{\mathrm{d}t}G_1(t, x)\right) + F_2(G_1(t, x)). \tag{16}$$

We note that while rewriting equation (15) in the form (16), we demand the function $F_2(t, x)$ in (15) automatically be a function of $G_1(t, x)$. Now identifying the function G_1 as the new dependent variable and the integral of G_2 over time as the new independent variable, that is,

$$w = G_1(t, x) \qquad z = \int_o^t G_2(t', x, \dot{x})\,\mathrm{d}t' \tag{17}$$

one indeed obtains an explicit transformation to remove the time-dependent part in the first integral (4). We note here that the integration on the right-hand side of (17) leading to z can be performed provided the function G_2 is an exact derivative of t, that is, $G_2 = \frac{\mathrm{d}}{\mathrm{d}t}z(t, x) = \dot{x}z_x + z_t$, so that z turns out to be a function of t and x alone. In terms of the new variables, equation (16) can be modified to the form

$$I = F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) + F_2(w).$$

In other words

$$F_1\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right) = I - F_2(w). \tag{18}$$

Now rewriting equation (18) one obtains a separable equation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(w) \tag{19}$$

which can be integrated by quadrature.

For the special case of the Duffing–van der Pol oscillator equation (6), by following the procedure given above, we can deduce the transformation (7) systematically from the first integral (4). For this purpose, we rewrite equation (5) as

$$\hat{I} = \frac{\beta}{2}\left(\dot{x} + \frac{1}{\beta}x\right)\mathrm{e}^{\frac{3}{\beta}t} + \frac{\beta^2}{6}x^3\,\mathrm{e}^{\frac{3}{\beta}t} \tag{20}$$

where $\hat{I} = \frac{\beta}{2}I$. Equation (20) can be further split into

$$\hat{I} = -\frac{\beta}{2}\,\mathrm{e}^{\frac{2}{\beta}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(-x\,\mathrm{e}^{\frac{1}{\beta}t}\right) + \frac{\beta^2}{6}\left(x\,\mathrm{e}^{\frac{1}{\beta}t}\right)^3. \tag{21}$$

Comparing equation (21) with (16) we get

$$G_1 = -x\,\mathrm{e}^{\frac{1}{\beta}t} \qquad G_2 = \frac{2}{\beta}\,\mathrm{e}^{-\frac{2}{\beta}t} \tag{22}$$

which in turn provides the transformation coordinates, w and z, through relation (17) of the form

$$w = -x\,\mathrm{e}^{\frac{1}{\beta}t} \qquad z = \mathrm{e}^{-\frac{2}{\beta}t}.$$

4. Applications

We observe that the above procedure can be extended to solve a class of equations. In the following we briefly describe the applicability of this method to the Duffing oscillator and the equation describing the motion of a gaseous general relativistic fluid sphere. A detailed connection between the integrating factors, integrals of motion and the transformation coordinates will be published elsewhere [6].

4.1. *Duffing oscillator*

Let us consider the Duffing oscillator [1]

$$\ddot{x} + \alpha\dot{x} + \beta x + x^3 = \gamma \cos\omega t \tag{23}$$

where α, β, γ and ω are arbitrary parameters. In the absence of the external forcing ($\gamma = 0$), equation (23) has been shown to be integrable for a specific parametric choice [7], namely, $2\alpha^2 = 9\beta$. The first integral for this parametric choice has been constructed in the form

$$I = \mathrm{e}^{\frac{4}{3}\alpha t}\left[\frac{\dot{x}^2}{2} + \frac{\alpha x\dot{x}}{3} + \frac{\alpha^2 x^2}{18} + \frac{x^4}{4}\right]. \tag{24}$$

Equation (24) is a complicated nonlinear first-order ordinary differential equation (ODE) with explicit time-dependent coefficients and so cannot be integrated directly. The usual way to overcome this problem is to transform equation (24) to an autonomous equation and integrate it. Using our procedure one can construct the required transformations systematically. For example, rewriting equation (24) in the form (15) we get

$$I = \frac{1}{2}\left(\dot{x} + \frac{\alpha x}{3}\right)^2 \mathrm{e}^{\frac{4}{3}\alpha t} + \frac{x^4}{4}\,\mathrm{e}^{\frac{4}{3}\alpha t}. \tag{25}$$

Now splitting the first term in equation (25) further in the form of (16),

$$I = \left[\mathrm{e}^{\frac{\alpha}{3}t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{\sqrt{2}}\,\mathrm{e}^{\frac{\alpha}{3}t}\right)\right]^2 + \left(\frac{x}{\sqrt{2}}\,\mathrm{e}^{\frac{\alpha}{3}t}\right)^4 \tag{26}$$

and identifying the dependent and independent variables from (26) using relation (17), we obtain the transformation

$$w = \frac{1}{\sqrt{2}}x\,\mathrm{e}^{\frac{\alpha t}{3}} \qquad z = -\frac{3}{\alpha}\,\mathrm{e}^{-\frac{\alpha t}{3}}. \tag{27}$$

One can easily check that equation (24) can be transformed into the autonomous form with the help of the transformation (27) and the latter can be integrated in terms of a Jacobian elliptic function [7]. We note that the transformation (27) exactly coincides with the one known in the literature which has been constructed in an *ad hoc* way.

4.2. *Static gaseous general relativistic fluid sphere*

Recently, Duarte *et al* [8] have constructed the first integral for an equation describing the static gaseous general relativistic fluid sphere [9],

$$\ddot{x} = \frac{t^2\dot{x}^2 + x^2 - 1}{t^2 x} \tag{28}$$

through the so-called Prelle–Singer procedure, in the form

$$I = \frac{2tx\dot{x} + x^2 + t^2\dot{x}^2 - 1}{t^2x^2}. \tag{29}$$

We note that unlike the earlier two examples, namely the Duffing and Duffing–van der Pol oscillators, we have no exponential function in the first integral. Also the first integral is a rational function. However, in the following we show that one can integrate it and obtain a general solution for this problem. By following our procedure, equation (29) can be recast in the form

$$I = \left(\frac{x + t\dot{x}}{tx}\right)^2 - \frac{1}{t^2x^2}. \tag{30}$$

One may note that the first term in (30) can be written in terms of a perfect differential form, that is, $\left(\frac{\mathrm{d}}{\mathrm{d}t}[\log(tx)]\right)^2$ so that $G_1 = \log(tx)$ and the function G_2 turns out to be a constant, $G_2 = 1$, in the present example. As a consequence we obtain a transformation

$$w = \log(tx) \qquad z = t \tag{31}$$

which can be utilized to rewrite equation (29) in the autonomous form

$$w'^2 - \exp(-2w) = I. \tag{32}$$

Equation (32) can be integrated under two different choices [10], namely, (i) $I > 0$ and (ii) $I < 0$. The respective solutions are

$$\begin{aligned} &\text{(i)} \quad z - z_0 = -\frac{1}{2\sqrt{I}} \log\left(\frac{\sqrt{I + \mathrm{e}^{-2w}} - \sqrt{I}}{\sqrt{I + \mathrm{e}^{-2w}} + \sqrt{I}}\right) \qquad I > 0 \\ &\text{(ii)} \quad z - z_0 = -\frac{1}{\sqrt{-I}} \arctan\left(\frac{\sqrt{I + \mathrm{e}^{-2w}}}{\sqrt{-I}}\right) \qquad I < 0 \end{aligned} \tag{33}$$

where z_0 is the second integration constant. Rewriting equation (33) we get

$$w = \begin{cases} \log\left(\dfrac{1 - \mathrm{e}^{-2\sqrt{I}(z-z_0)}}{2\sqrt{I}\mathrm{e}^{-\sqrt{I}(z-z_0)}}\right) & I > 0 \\ \log\left(\dfrac{\cos\sqrt{-I}(z - z_0)}{\sqrt{-I}}\right) & I < 0. \end{cases} \tag{34}$$

Utilizing (31) in (34) one can write down the solution for the static gaseous general relativistic fluid sphere of the form

$$x = \begin{cases} \dfrac{1}{\sqrt{I}t} \sinh\sqrt{I}(t - t_0) & I > 0 \\ \dfrac{1}{\sqrt{-I}t} \cos\sqrt{-I}(t - t_0) & I < 0. \end{cases} \tag{35}$$

To our knowledge this solution is new for this problem.

5. Linearization

Interestingly, our method not only helps to integrate the first integral and gives the complete solution for a class of second-order nonlinear ODEs but it also helps to deduce the transformations which can be used to linearize the given nonlinear ODEs in a very systematic way. For example, let us consider the modified Emden equation,

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0 \tag{36}$$

where α and β are arbitrary parameters. Mahomed and Leach [11] have shown that equation (36) is linearizable through point transformations for a particular parametric choice, namely, $\alpha^2 = 9\beta$. The linearizing transformation and the first integral are known to be

$$w = \frac{t}{x} - \frac{\alpha t^2}{6} \qquad z = \frac{\alpha}{3}t - \frac{1}{x} \tag{37}$$

and

$$I = -t + \frac{x}{\frac{\alpha}{3}x^2 + \dot{x}}. \tag{38}$$

Substituting (37) into (36) one can transform the latter into the free particle equation, that is

$$\frac{d^2 w}{dz^2} = 0. \tag{39}$$

The authors have derived the linearizing transformation through Lie symmetry analysis [11]. However, in the following we show that the linearizing transformation (37) can also be derived from the first integral itself through our method.

Rewriting the first integral (38) in the form

$$I = \frac{x^2}{\frac{\alpha}{3}x^2 + \dot{x}} \left[\frac{d}{dt} \left(\frac{t}{x} - \frac{\alpha t^2}{6} \right) \right] \tag{40}$$

and identifying (40) with (16) we get

$$G_1 = \frac{t}{x} - \frac{\alpha t^2}{6} \qquad G_2 = \frac{\alpha}{3} + \frac{\dot{x}}{x^2} \qquad F_2 = 0. \tag{41}$$

One can see that, unlike the earlier examples discussed so far, in the present example G_2 depends both on x and $\dot{x}$. However, as we noted earlier the new variable z can be obtained once the function G_2 is a perfect derivative of t, that is,

$$G_2 = \frac{\alpha}{3} + \frac{\dot{x}}{x^2} \equiv \frac{d}{dt} \left(\frac{\alpha}{3} t - \frac{1}{x} \right) \tag{42}$$

so that (17) gives

$$w = \frac{t}{x} - \frac{\alpha t^2}{6} \qquad z = \frac{\alpha}{3} t - \frac{1}{x}$$

which is nothing but the linearizing transformation. One may note that in this case while rewriting the first integral I (equation (38)) in the form (15), the function F_2 disappears, that is $F_2 = 0$, and as a consequence we arrive at (see equation (18))

$$\frac{dw}{dz} = I \tag{43}$$

which in turn gives (39) by differentiation or leads to the solution by an integration. On the other hand, vanishing of the function F_2 in this analysis is precisely the condition for the system to be transformed into the free particle equation [6].

6. Conclusion

One of the common problems in solving second-order nonlinear ODEs is how to solve the time-dependent first integral and obtain a general solution associated with the given equation. In fact most of the methods available to tackle the second-order nonlinear ODEs provide first integrals only. In this work, we have proposed a novel method for identifying transformation coordinates which can be used to rewrite the first integral without explicit time dependence so that it can be integrated by quadrature. Interestingly, we showed that the transformation coordinates can be constructed from the first integral itself. With this choice, the above procedure can be augmented with other existing methods to obtain a complete solution for a given problem. We have also shown the applicability of our method in constructing linearizing transformations as well in a simple and straightforward way. Finally, we mention that the proposed method can also be used to study certain coupled nonlinear oscillators, the details of which will be presented separately [6].

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